A: Brief Review of Ordinary Differential Equations

Because of Principle # 1 mentioned in the Opening Remarks section, you should review your notes from your ordinary differential equations (odes) course and keep your old textbook handy for reference. The comments here are *not* meant to be a substitute, but rather as a guide to the most important parts of odes relevant to our pde course.

For our purposes we are only concerned with solving linear first-order and linear second-order odes. In your first course you probably only discussed initial value problems, that is, time-dependent problems defined for $t > t_0$. The new topic in odes we need for this pde course is to solve (two-point) *boundary value problems* (bvps). For now we want you to review some specific aspects of initial-value problems (ivps).

Unlike pdes, odes can be classified more or less by order. The *order of an ode* is the order of the highest derivative appearing in the equation.

1 Linear First-Order ODEs

These equations have the form $A\frac{dy}{dt} + By + C = 0$, where A, B, C can depend at most on the variable t, and we assume $A \neq 0$ either as a constant, or as a function of t. For precise conditions to have the existence of a global solution, consult any ODE book. If we assume A never vanishes on the interval of interest, let $p = p(t) := B/A$, $q = q(t) := -C/A$, then we can write the equation in the standard form

$$\frac{dy}{dt} + p(t)y = q(t) \quad . \tag{1}$$

Examples: (a), (b), (c) are linear equations, (d) and (e) are not linear.
(a) $\frac{dy}{dt} = \sin(2t - 1)$

(b) $(1 + t^2)\frac{dy}{dt} + \ln(t)y = e^t$

(c) $\sin(2t)dy + \cos(2t)dt = 0$

(d) $4\frac{dy}{dt} + 3y^2 = tanh(t)$

(e) $y\frac{dy}{dt} + ty + t^2 = 0$

Remark: Given a first-order equation, realize that it takes an integration to solve it, so the general solution to a first-order equation involves a constant of integration. That is, without any extra constraints imposed on the solution besides the equation, the solution is a 1-parameter family of functions.

Example: $\frac{dy}{dt} = 1 \to y(t) = t + C$, while $\frac{dy}{dt} + \sin(t)y = 0 \to y(t) = Ce^{\cos(t)}$. Thus, a *well-posed problem* includes the equation, the domain over which it holds, and a value for the solution at a given point. For example, here is a well-posed problem:

$$\frac{dy}{dt} + \sin(t)y = 0 \ , \quad t > 0, \quad y(0) = 1 \ .$$

Therefore, $y(t) = e^{-1+\cos(t)}$ is the solution for $t > 0$. (*Exercise:* Show this.)

Note that the equation need not hold at the "initial" point $t = 0$ because we do not require the solution to have a derivative there (though in this example the derivative does exist and is continuous on $[0, \infty)$).

First order linear equations should be solved by the *method of integrating factors*. Given (1), integrate $p(t)$, then exponentiate the result. The resulting expression is the **integrating factor** (IF). For example, in the example above, $e^{-\cos(t)}$ is the IF. The IF for the general equation (1) is $e^{\int_0^t p(s)ds}$. Multiplying the equation by the IF, you can combine the two terms on the left side of the equation into and exact derivative. For equation (1) we would have

$$\frac{d}{dt}(ye^{\int_0^t p(s)ds}) = q(t)e^{\int_0^t p(s)ds} \quad .$$

For the specific example

$$\frac{d}{dt}(ye^{-\cos(t)}) = 0 \ .$$

Hence,

$$ye^{-\cos(t)} = \text{constant} = C \quad \to \quad y(t) = Ce^{\cos(t)}$$

$$y(t)_{|t=0} = 1 \quad \to \quad 1 = Ce^1 \quad \to \quad y(t) = e^{-1+\cos(t)}$$

Example: $\frac{dy}{dt} + py = q$, where p, q are constants.

IF $= e^{pt}$, so $e^{pt}\frac{dy}{dt} + pe^{pt}y = qe^{pt}$, or $\frac{d}{dt}(e^{pt}y) = qe^{pt}$, hence $e^{pt}y(t) = \frac{q}{p}e^{pt} + C$, which implies that $y(t) = \frac{q}{p} + Ce^{-pt}$.

Comment: If you know the domain of the problem it is best to use that information by performing **definite integration** rather than indefinite integration. For example, if in the last example we have it hold for $t > 0$, and $y(0) = 2$, then write $\frac{d}{dt}(e^{pt}y) = qe^{pt} \quad \rightarrow \quad e^{pt}y(t) - y(0) = \int_0^t qe^{pr}dr$, or $e^{pt}y(t) - 2 = \frac{q}{p}e^{pr}|_0^t = \frac{q}{p}(e^{pt} - 1)$, or $y(t) = 2e^{-pt} + \frac{q}{p}(1 - e^{-pt})$. Assuming $p > 0$, one clearly sees $y(0) = 2$ and that $y(t) \rightarrow q/p$ as $t \rightarrow \infty$.

Consider

$$\frac{dy}{dt} - 2ty = 1 \quad , \quad y(0) = 3, \text{where the domain is } t > 0 \ .$$

The IF is e^{-t^2}. Therefore, $\frac{d}{dt}(e^{-t^2}y) = e^{-t^2}$, so $e^{-t^2}y(t) - 3 = \int_0^t e^{-r^2}dr \quad \rightarrow$ $y(t) = 3e^{t^2} + e^{t^2}\int_0^t e^{-r^2}dr$.

Remark: When integrating a definite integral with a variable limit of integration, use a "dummy" variable of integration, here denoted by r (arbitrarily). That is, $e^{-t^2}y(t) - 3 = \int_0^t e^{-t^2}dt$ is **wrong**. Remember that t is a variable limit of integration, so you must use a different variable of integration.

Remark: A special function that comes up often in pdes (particularly heat and mass transfer), and other areas of mathematics, is the **error function**. It is defined as

$$erf(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds \quad .$$

Note the following properties:
(i) For $z > 0$, $erf(z) > 0$;
(ii) $erf(0) = 0$;
(iii) $\frac{d}{dz}erf(z) = \frac{2}{\sqrt{\pi}}e^{-z^2} > 0$, so erf is a monotone increasing function;
(iv) $\int_0^\infty e^{-s^2}ds = \frac{\sqrt{\pi}}{2}$, so $erf(z) \rightarrow 1$ as $z \rightarrow \infty$;
(v) If $z < 0$, $erf(z) = -erf(-z)$, that is, erf is an odd function.

Therefore, the solution to the last example could be written as $y(t) = 3e^{t^2} + \frac{\sqrt{\pi}}{2}e^{t^2}erf(t)$. (We'll run across the error function again in the pde Notes.)

Exercises for practice

1. $\frac{dy}{dt} + 3y = 0$
2. $t\frac{dy}{dt} + 4y = t^2$ on any interval not containing $t = 0$
3. $\sin(t)\frac{dy}{dt} + \cos(t)y = \cos(t)$, for $\pi/2 < t < \pi$
4. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 2$
   Note that you can integrate directly the expression to get a first-order equation to solve.

(Answers: $y(0) = y(0)e^{-3t}$; $y(t) = t^2/6 + C/t^4$; $y(t) = 1 + C\csc(t)$; $y(t) = \frac{1}{2}t + C_1e^{-4t} + C_2$)

2 Linear Second-Order ODEs

These are of the form $Ay'' + By' + Cy + D = 0$, with $A \neq 0$, but we most commonly write the equation in the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t) \tag{2}$$

where the coefficients are generally continuous on some interval (t_0, t_1), and $a(t) \neq 0$ in the interval. The nonhomogeneous function f may be a sum of terms, but if $f \equiv 0$, the equation (2) is **homogeneous**, otherwise the equation is **nonhomogeneous**. For variable coefficients a, b, c there is no general procedure for obtaining a formula for the general solution, even though we know the existence of solutions, and a general form for them. Of course, if we have one solution, we can obtain another one by the reduction of order method that you can look up in any ODE book.

So the strategy for equation (2), with the coefficients defined on some interval, say $t > 0$, is to first find two *linearly independent* solutions $y_1(t)$ and $y_2(t)$. Hence, y_1, y_2 are solutions to (2) such that the *Wronskian* $W(y_1, y_2)(t) :=$

$y_1(t)y_2'(t) - y_2(t)y_1'(t) \neq 0$ for any t in the interval. With these requirements, $y_1(t)$ and $y_2(t)$ form a **fundamental set of solutions** for (2), and the general solution to (2) is given be their linear combination: $y(t) = C_1 y_1(t) + C_2 y_2(t)$.

Example: Consider $\frac{d^2y}{dt^2} - 4y = 0$ on the real line. Then e^{2t}, e^{-2t} form a fundamental set of solutions ($W(e^{2t}, e^{-2t}) \equiv -4$ for all t). But so is $sinh(2t), cosh(2t)$, and $e^{2t}, cosh(2t)$, so fundamental set of solutions to a given equation is not unique.

Needed for this level of pdes are two classes of problems, the constant coefficient equations, and the Cauchy-Euler equations. (These two classes of equations are actually equivalent, but we will not make use of that fact.)

2.1 Constant-coefficient equations

Consider (2) with a, b, c being real constants, and $f(t) \equiv 0$. Assume a solution of the form $y(t) = e^{rt}$, and substitute this into the equation. Then you obtain

$$e^{rt}\{ar^2 + br + c\} = 0 \tag{3}$$

since e^{rt} is never zero, the quadratic in the brackets must be zero. That is, its roots are the only values of r for which $y(t) = e^{rt}$ is a solution. Thus, $ar^2 + br + c = 0$ is the **characteristic equation** of the differential equation and

$$y(t) = e^{rt} \quad \text{is a solution to (2) if, and only if} \quad r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4}$$

There are three non-degenerate possibilities regarding (4):

1. roots r_1, r_2 are real, unequal: then a fundamental set of solutions is $e^{r_1 t}, e^{r_2 t}$, so the general solution is $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

2. roots r_1, r_2 are real, equal (say $r := r_1 = r_2$): then a fundamental set of solutions is e^{rt}, te^{rt}, so the general solution is $y(t) = \{C_1 + C_2 t\}e^{rt}$

3. the roots are complex conjugates, say $r_{1,2} = a \pm bi$, with $b \neq 0$ ($i = \sqrt{-1}$): then a fundamental set of solutions is $e^{at}\cos(bt), e^{at}\sin(bt)$, and the general solution is $y(t) = e^{at}\{C_1 \cos(bt) + C_2 \sin(bt)\}$.

Example: Mass-spring model $m\frac{d^2y}{dt^2} + ky = 0$
Let $\omega := \sqrt{k/m}$, then $\frac{d^2y}{dt^2} + \omega^2 y = 0$. ($\omega$ has units of 1/time, so it is called the *natural frequency of the system*. Now $y(t) = e^{rt} \to r^2 + \omega^2 = 0 \to r = \pm i\omega$. Thus, from the third case, $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$. From the addition formulas, we can also write the solution in the form $y(t) = A\cos(\omega t - \phi)$, where now A is the *amplitude* of the motion (y oscillates between $-A$ and A), and ϕ is called the *phase* of the motion. These two numbers uniquely characterize the motion if we are given initial conditions.

Example Spring-mass-dashpot model
By including frictional forces, we have

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \quad t > 0$$

$$y(0) = 2, \frac{dy}{dt}(0) = 0$$

Thus, we pull down the mass 2 units of length and release it. Now substituting in $y(t) = e^{rt}$, we obtain for the characteristic equation $mr^2 + cr + k = 0$. Consider the mass, m, and spring constant, k, as being fixed, but that our *damping constant* c being able to take on different values. The possibilities are ($r_{1,2} = \frac{1}{2m}\{-c \pm \sqrt{c^2 - 4km}\}$)

1. roots are real, negative (discriminant $= c^2 - 4km > 0$)
   Thus

   $$y(t) = C_1e^{r_1t} + C_2e^{r_2t} = e^{-ct/2m}\{C_1e^{t\sqrt{c^2-4km}/2m} + C_2e^{-t\sqrt{c^2-4km}/2m}\}$$

   Given the two initial conditions and solving for the two constants, we arrive at

   $$y(t) = 2\{\frac{r_1}{r_1 - r_2}e^{r_1t} - \frac{r_2}{r_1 - r_2}e^{r_2t}\}$$

2. roots are real and equal (and negative: i.e. $c^2 - 4km = 0$, so $r_1 = r_2 = -c/2m$). Therefore,

   $$y(t) = e^{-ct/2m}\{C_1 + C_2t\} \to y(t) = e^{-ct/2m}\{2 + ct/m\}$$

3. roots are complex conjugates (i.e. $c^2 - 4km < 0$, so if we let $\beta := \sqrt{4km - c^2}/2m$, then $r_{1,2} = -\frac{c}{2m} \pm i\beta$). Now

   $$y(t) = e^{-ct/2m}\{C_1\cos(\beta t) + C_2\sin(\beta t)\}$$

After applying the initial conditions, we have

$$y(t) = e^{-ct/2m}\{2\cos(\beta t) + \frac{2c}{\sqrt{4km - c^2}}\sin(\beta t)\} = 4\sqrt{\frac{km}{4km - c^2}}\cos(\beta t - \phi)$$

where the phase $\phi = arctan(\frac{c}{2m\beta})$.

In case 1 both terms decay exponentially and this is termed the *overdamped* case (decay, but no oscillations; damping parameter c can be considered "large"). Case 3 has damped oscillation behavior (c is "small"), so is termed the *underdamped* case. Case 2 is an intermediate case called the *critically damped* case.

Practice Problems

1. $y'' + 4y = 0$
2. $y'' + 3y' = 0$
3. $y'' = 0$, $t > 0$, $y(0) = 0, y'(0) = 4$
4. $3y'' - 5y' - 2y = 0$
5. $y'' + 4y' = 2$, $t > 0$, $y(0) = 0, y'(0) = 4$

2.2 Cauchy-Euler Equations

These are of the type

$$\begin{aligned} &At^2\tfrac{d^2y}{dt^2} + Bt\tfrac{dy}{dt} + Cy = 0 \qquad \text{or} \\ &A(x - x_0)^2\tfrac{d^2y}{dx^2} + B(x - x_0)\tfrac{dy}{dx} + Cy = 0 \end{aligned} \tag{5}$$

where A, B, C are constants. (The latter equation can be transformed to the first equation by the transformation $t = x - x_0$.) It is assumed the first equation (respectively, the second equation) is defined on an interval not containing $t = 0$ ($x = x_0$). Then let $y = t^r$ (respectively, $y = (x - x_0)^r$), substitute into the equation to obtain

$$At^2(r(r-1)t^{r-2}) + Bt(rt^{r-1}) + Ct^r = t^r\{Ar^2 + (B - A)r + C\} = 0$$

so

$$Ar^2 + (B - A)r + C = 0 \tag{6}$$

is the **characteristic equation** for the Cauchy-Euler equation (5). Again we have three cases to consider:

case 1: roots r_1, r_2 are real, unequal.
Then a fundamental set of solutions is t^{r_1}, t^{r_2}, and the general solution to (5) is $y(t) = C_1 t^{r_1} + C_2 t^{r_2}$.
example: $t^2 y'' + 12ty' + 30y = 0$. Then $r = -5, -6$, so the general solution is $y(t) = C_1 t^{-5} + C_2 t^{-6}$.

case 2: roots r_1, r_2 are real and equal.
Let $r = r_1 = r_2$, then a fundamental set of solutions is $t^r, t^r \ln(|t|)$. Say the domain of equation (5) is $\mathbb{R}^+$ (or a subset of it), then the general solution to (5) is $y(t) = t^r\{C_1 + C_2 \ln(t)\}$.
example: $(t-4)^2 y'' + 7(t-4)y' + 9y = 0$ on domain $t > 4$. Then letting $y(t) = (t-4)^r$, we obtain roots to (6) being $r = -3, -3$, so the general solution is $y(t) = (t-4)^{-3}\{C_1 + C_2 \ln(t-4)\}$.

case 3: roots r_1, r_2 are complex conjugates.
Write $r_{1,2} = a \pm ib$, $b \neq 0$. Again assume the domain is in $\mathbb{R}^+$. Then a fundamental set of solutions for (5) is $t^a \cos(b \ln(t)), t^a \sin(b \ln(t))$, and the general solution is given by $y(t) = t^a\{C_1 \cos(b \ln(t)) + C_2 \sin(b \ln(t))\}$.
example: $(x+1)^2 y'' - \frac{1}{2}y = 0$, $x > 0$. Then the roots are $r = \frac{1}{2}(1 \pm \sqrt{3})$, so the general solution is $y(x) = \sqrt{x+1}\{C_1 \cos(\frac{\sqrt{3}}{2} \ln(x+1)) + C_2 \sin(\frac{\sqrt{3}}{2} \ln(x+1))\}$.

2.3 Non-homogeneous 2nd order linear equations

Given the general equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t) \qquad t > a \tag{7}$$

with $f(t) \neq 0$, then if (7) has two solutions, say y_1, y_2, then $u(t) = y_1(t) - y_2(t)$ satisfies the homogeneous version of (7), which can be seen by simply writing (7) with y_1 and y_2, and subtracting the two equations; so $u'' + p(t)u' + q(y)u = 0$. Since this equation has a fundamental set of solutions u_1, u_2, then

$u(t) = C_1u_1(t)+C_2u_2(t)$. Hence, $y_1(t) = C_1u_1(t)+C_2u_2(t)+y_2(t)$. This is the solution to (7) with two free parameters to satisfy initial conditions (or two boundary conditions), so it represents the general solution to the homogeneous equation plus a particular solution to the non-homogeneous equation. Thus, given (7) along with initial conditions (or boundary conditions on a finite interval), the solution strategy is (order is paramount here)

1. Find a fundamental set of solutions $u_1(t), u_2(t)$ to the homogeneous equation.
2. Find a particular solution, $y_p(t)$, to the non-homogeneous equation.
3. Write the general solution to the (non-homogeneous) equation as $y(t) = C_1u_1(t)+C_2u_2(t)+y_p(t)$, and apply the initial conditions (respectively, boundary conditions), to determine the unique solution to the problem.

Note: The order is important here because to enact step 2 you need information about the fundamental set of solutions to the homogeneous equation in step 1.

There are two main approaches to obtaining a particular solution. they are (i) method of undetermined coefficients, and (ii) variation of parameters method.

2.4 Method of undetermined coefficients

Most of the odes encountered in these pre Notes will be constant coefficient equations, so the undetermined coefficients method is most useful for us. But is is more restrictive than variation of parameters method. However, when it is applicable, it is generally easier to apply than the general method of variation of parameters. Undetermined coefficient method requires the coefficients on the left-hand side of (7) to be *constant coefficients*, and the right-hand side of the equation, the non-homogeneity $f(t)$, to be of a special form, namely to be of the form

$$e^{at} \times P(t) \times \begin{cases} \cos(bt) \\ \sin(bt) \end{cases} \tag{8}$$

or sums of such terms as these (where $P(t)$ is a given polynomial in t). What is meant by (8) is that not every term need to be represented. The sin and/or

cos could be missing (and $P(t) \equiv 1$), so the term would be a simple exponential, or the the trig functions could be missing, and $a = 0$, so the term would be just a polynomial, etc. Examples of such valid right-hand sides that the method could be applied to would be
$f(t) = 2 - 5\cos(3t)$
$f(t) = e^{-10t}\sin(t)$
$f(t) = 15cosh(3t - 7) + 18t^3 - 1$
etc.
and examples that the method of undetermined coefficients method does **not** apply (hence, one must use the method of variation of parameters) would be
$f(t) = sec(2t)$
$f(t) = 5\ln(2t)$
$f(t) = t^2 + 1/t^2$.

Example: Consider

$$\frac{d^2y}{dt^2} + \omega^2 y = 3e^{-t}\cos(2t) \ , \ y(0) = 0 \ , \ \frac{dy}{dt}(0) = 1 \ .$$

A fundamental set of solutions for $y'' + \omega^2 y = 0$ is $\sin(\omega t), \cos(\omega t)$. For the non-homogeneity $f(t) = 3e^{-t}\cos(2t)$, let $y_p(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$. Then ask: Is any term here a solution to the homogeneous equation? If the answer were yes, then multiply the right side of the expression for y_p by t and ask the question again. If the answer were still yes, you would have to multiply by another t, then proceed to substitute the form of y_p into the equation. However, for this specific example the answer is no and so we use the original expression in the non-homogeneous equation. Thus

$$e^{-t}\{(\omega^2 - 3)A - 4B\}\cos(2t) + e^{-t}\{(\omega^2 - 3)B + 4A\}\sin(2t) = 3e^{-t}\cos(2t)$$

so

$$A = \frac{3(\omega^2 - 3)}{(\omega^2 - 3)^2 + 16} \ , \ B = -\frac{12}{(\omega^2 - 3)^2 + 16} \ .$$

Thus, the general solution to the problem is

$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) \ .$$

Now $y(0) = 0 = C_1 + A, y'(0) = 1 = \omega C_2 - A + 2B$. Solving for C_1, C_2, we have

$$y(t) = \frac{1}{\omega}(1 - 2B + A)\sin(\omega t) - A\cos(\omega t) + Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) \ ,$$

where A, B are given above.

It is now up to you to pick up your ODE book and go through enough examples that you feel comfortable solving such problems. You need to be comfortable solving odes to succeed in pdes at this level.

2.4.1 Variation of parameters method

We will demonstrate this method through a couple of examples. It is your responsibility to look up more examples in your ODE book.

Example: Consider

$$\frac{d^2y}{dt^2} + y = 2\sec(t) \qquad 0 < t < \pi/2$$

The equation has constant coefficients, but the right-hand side does not have the form of non-homogeneity that allows us to use undetermined coefficients method. A fundamental set of solutions to the homogeneous equation is $\cos(t), \sin(t)$, so for a particular solution to the non-homogeneous equation, let $y_p(t) = u(t)\cos(t) + v(t)\sin(t)$. Then

$$\frac{dy_p}{dt} = \frac{du}{dt}\cos(t) - u\sin(t) + \frac{dv}{dt}\sin(t) + v\cos(t) \ .$$

Set

$$\frac{du}{dt}\cos(t) + \frac{dv}{dt}\sin(t) = 0 \ . \tag{9}$$

This prevents having second derivatives in the unknown functions u, v when taking a second derivative of y_p. Now

$$\frac{d^2y_p}{dt^2} = -\frac{du}{dt}\sin(t) - u\cos(t) + \frac{dv}{dt}\cos(t) - v\sin(t) \ .$$

Now add the expression for y_p:

$$\frac{d^2y_p}{dt^2} + y_p = -\frac{du}{dt}\sin(t) + \frac{dv}{dt}\cos(t) = 2\sec(t) \ . \tag{10}$$

From (9) we have $u' = -v'\tan(t)$, so substituting this into (10) gives

$$\frac{d^2y_p}{dt^2} + y_p = -(-\frac{dv}{dt}\tan(t))\sin(t) + \frac{dv}{dt}\cos(t) = 2\sec(t) \ .$$

Multiply by $\cos(t)$ we obtain

$$\frac{dv}{dt}(\sin^2(t)+\cos^2(t)) = \frac{dv}{dt} = 2 \quad \rightarrow \quad v = 2t$$

and

$$\frac{du}{dt} = -2\tan(t) \quad \rightarrow \quad u = -ln(|\cos(t)|) = -ln(\cos(t))$$

hence $y_p = -\cos(t)ln(\cos(t)) + 2t\sin(t)$, so the general solution to the equation is

$$y(t) = C_1\cos(t) + C_2\sin(t) - \cos(t)ln(\cos(t)) + 2t\sin(t) \quad \text{for } 0 < t < \pi/2\ .$$

Example: Consider

$$t^2\frac{d^2y}{dt^2} + 12t\frac{dy}{dt} + 30y = 3t^2 \qquad 0 < t$$

Now, as a variable coefficient equation on the left-hand side, the undetermined coefficients method is not applicable. A fundamental set of solutions for the homogeneous (Cauchy-Euler) equation is t^{-5}, t^{-6}, so let $y_p(t) = t^{-5}u(t) + t^{-6}v(t)$. Again take a derivative and set $t^{-5}u'(t) + t^{-6}v'(t) = 0$,as done in (9), so that $v' = -tu'$. Then $y_p^{'}(t) = -5t^{-6}u - 6t^{-7}v$ and so $y_p^{''}(t) = 30t^{-7}u + 42t^{-8}v - 5t^{-6}u' - 6t^{-7}v'$, so after substituting into the equation, we have

$$t^2y_p^{''} + 12ty_p^{'} + 30y_p = -5t^{-4}u' - 6t^{-5}v' = 3t^2$$

or substituting for v',

$$-5t^{-4}u' - 6t^{-5}(-tu') = t^{-4}u' = 3t^2 \quad \rightarrow \quad u' = 3t^6 \quad \rightarrow \quad u = \frac{3}{7}t^7\ .$$

This gives $v' = -3t^7$, or $v = -\frac{3}{8}t^8$, so $y_p = \frac{3}{7}t^2 - \frac{3}{8}t^2 = \frac{3}{56}t^2$. Finally, the general solution to the equation is

$$y(t) = C_1t^{-5} + C_2t^{-6} + \frac{3}{56}t^2\ .$$

3 Exercises

1. $t\frac{d^2y}{dt^2} + \frac{dy}{dt} = -1$, $t > 0$, $y(1) = 0, y'(1) = 0$.

2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 3e^{-x}$

3. If $u_1(x) = 1 + x, u_2(x) = e^x$ form a fundamental set of solutions to the homogeneous equation, what is the general solution to

$$x\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} + y = x^2e^{2x} \quad x > 0$$

4. What is the characteristic equation for

   (a) $2y'' - 2y' + y = 0$

   (b) $(1+x)^2y'' + (1+x)y' - 16y = 0$

   Write a fundamental set of solutions for each equation.

5. If $u_1(x) = e^{2x}, u_2(x) = xe^{2x}$ are two solutions to the equation $y'' - 4y' + 4y = 0$, what is the Wronskian of these two functions and what does it say about the nature of these solutions?

6. Define or characterize the following:

   (a) error and complimentary error functions

   (b) fundamental set of solutions

   (c) Wronskian of two functions

   (d) Cauchy-Euler equation

Some answers:
1. $y = 1 - t + ln(t)$
2. $y = C_1e^{-x} + C_2xe^{-x} + \frac{3}{2}x^2e^{-x}$
3. $y = C_1(1+x) + C_2e^x + \frac{1}{2}e^{2x}(x-1)$
4a. $e^{x/2}\cos(x/2), e^{x/2}\sin(x/2)$ 4b. $(1+x)^4, (1+x)^{-4}$